ENERGY-LEVEL CASCADES IN PHYSICAL SCALES OF 3D INCOMPRESSIBLE MAGNETOHYDRODYNAMIC TURBULENCE.

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ABSTRACT. The total energy in 3D MHD turbulence is shown to cascade over an inertial range the lower limit of which is a modified Taylor length scale. Direct cascades are also investigated for energies transported by distinct mechanisms which together comprise the total energy. These include the convection of total fluid energy by the velocity field, the kinetic to kinetic energy transfer advected by the velocity field, and the magnetic to magnetic energy transfer advected by the velocity field. Flux locality for the cascading quantities is also addressed. A scenario in which the inter-medium energy transfer is predominantly from kinetic to magnetic and a note regarding the cascade of kinetic energy for *non-decaying* fluid turbulence are included.

1. INTRODUCTION.

Turbulence remains an enigmatic feature of modern fluid and plasma dynamics. In the case of 3D incompressible fluids, the first positive rigorous result pertaining to the existence of energy cascades was achieved in Fourier space by Foias, Manley, Rosa, and Temam [17]. A pioneering step was recently taken in [10] where a mathematical apparatus was developed to study turbulence working entirely in *physical scales*. In this setting it was shown that ensemble averages of localized space-time averages (localized to certain coverings of an integral domain) of the energy flux through shells are positive and nearly constant. Consequently, the transport of energy is, in an averaged sense, directed from larger to smaller scale structures. Interestingly, *locality of the flux* in this context is naturally affirmed. More precisely, the locality is derived *dynamically* as a direct consequence of existence of the turbulent cascade in view, featuring comparable upper and lower bounds throughout the inertial range; in contrast, the previous locality results were essentially localized *kinematic* upper bounds on the flux, the corresponding lower bounds being consistent with turbulent properties of the flow [16, 8]. The formal similarities between the magnetohydrodynamical system and the Navier-Stokes equations allow the theory of the former to be informed by that of the latter. The present work proceeds in this spirit. Here, the methodology of Dascaliuc and Grujić is applied to establish the existence and locality of various energy-level cascades in turbulent MHD flows over suitable inertial ranges. An additional comment is included regarding preferential inter-field energy transport. A complementary work by the authors [6] is in preparation and will deal with *kinetic and magnetic enstrophy cascades* in 3D incompressible MHD. The two works combined will constitute a solid foundation to the theory of 3D MHD turbulent cascades/transport in physical scales.

Magnetohydrodynamical turbulence research has been energized by recent solar wind observations as well as the ability to generate more complex and more accurate numerical models. The picture painted by these advances is, however, far from conclusive. The most recent theories indicate energy cascades in certain settings and posit power-laws with dependencies on the viscosity and resistivity, as well as global quantities associated with the flow [3, 22, 4, 21, 19]. The first purpose of this paper is to rigorously establish the cascade of total energy by studying the total

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energy flux in a suitably statistical manner across a range of scales – the *inertial range* – for which the lower bound is a corrected *Taylor micro-scale*, the correction of which is dependent on parameters arising from the employed mathematical apparatus as well as the *magnetic Prandtl number*. This result follows directly from the velocity-magnetic field formulation of the 3D incompressible magnetohydrodynamic equations and is physically reasonable for all forms of MHD turbulence where the magnetic Prandtl number is not significantly smaller than one (i.e. $\eta \lesssim \nu$). In particular, it applies in astronomical settings such as the Solar wind and the interstellar medium where η is often so small that the non-resistive limit is commonly used as an approximation [3]. Interestingly, in contrast to current numerical and phenomonological theories, no appeal is made to the existence of a strong magnetic guide field and we thus assert that even in the absence of a strong background magnetic field, 3D MHD turbulence exhibits an inertial range over which a cascade of total energy transpires.

Due to the coupling between the magnetic and kinetic fields, each of which is imbued with its own energy, there are a number of transfer mechanisms by which energy can 'flow' between scales. In particular, energy can remain tied to the kinetic field or the magnetic field but it can also be transferred between the two fields–i.e. from the kinetic field to the magnetic field or vice versa. The secondary purpose of this work is to identify conditions under which the distinct energy transfers are, in an ensemble averaged sense, directed from larger to smaller scales. Particular attention is paid to the term responsible for the transfer of energy locked within the velocity field. This is the total fluid energy flux. Interestingly, the numerically and observationally supported phenomenon of *dynamic alignment* – the scale dependent depletion of the angle between the direction of the velocity and magnetic fields [3, 22, 4, 21] – can be interpreted to obtain a (weak) geometric depletion of the non-linearity complementary to the fluid energy flux. We investigate the role this plays in the relevant cascade.

In [13], the ensemble averaging mechanism which enables our study of turbulence in physical scales was interpreted to study non-flux type quantities (the vortex stretching term) and establish a mathematical evidence for the persistence of the lengths of vortex-filaments across a (longitudinal) range of scales. A similar argument can be applied to establish a scenario in which the inter-field energy exchange is predominantly oriented from the velocity field to the magnetic field. To achieve this we provide a dynamic estimate which entails the stretching effect of the velocity field on the magnetic field lines is predominantly one of elongation (the magnetic field is energized at the expense of the velocity field) as opposed to diminution (in which case the velocity field would be energized).

It is important to mention that pursuing a rigorous study of turbulent cascades in physical scales is a break from the traditional approach [9, 18]. Despite the dependence on properties of turbulence which are emergent in *physical space* present in the Kolmogorov dimensional analysis, the predominant setting for the study of these cascades has been in Fourier space, i.e., in the wavenumbers; this required postulating (in a statistical sense) isotropy and homogeneity of the flow [17, 16, 8]. A *direct physical interpretation* of scale as a platform to understand cascades has traditionally been rejected. The crucial problem for a physical interpretation is that the cascade does not occur locally – the energy flux cannot be expected to be, in an averaged sense, inwardly oriented on *any* particular shell. In the present framework, this is overcome by an ensemble averaging process whereby any local detractions from the essential inward-ness of the energy flux are shown to be insignificant and thus, in a statistically significant sense, the net 'flow' of energy is from larger to smaller scales.

Several subjects treated in the present manuscript are noteworthy in the context of the existing literature regarding fluid and plasma turbulence in physical scales. First is the development of

cascades for fluxes which exclude the pressure flux. Because analysis of the pressure flux is fundamentally different than other fluxes appearing in both MHD and NSE, additional steps must be carried out to interpolate ensemble averages of localized quantities between integral scale quantities. Our present treatment of the fluid pressure flux provides an avenue for the study of *non-decaying* turbulence which will be investigated in a future work. Second, the relationship between cascades for the dimensional and dimensionless forms of the equations is investigated and found to be equivalent. This is technically beneficial because, by carrying out analysis after passing to the dimensionless form, issues of dimensionality are tidily avoided.

2. (K_1, K_2)-Covers and Ensemble Averages.

The main purpose of this section is to describe how *ensemble averaging* with respect to (K_1, K_2)-*covers* of an integral domain $B(0, R_0)$ can be used to establish *essential positivity* of an *a priori* sign-varying density over a range of physical scales associated with the integral domain (cf. [10, 11, 14, 12]). The application to turbulence is establishing the positivity of certain inward directed flux densities – i.e. the cascade is uni-directional from larger to smaller scales – as well as the near-constancy of the averaged densities – i.e. the space-time averages over cover elements are all mutually comparable – across a range of scales.

The ensemble averages will be taken over collections of spatio-temporal averages of physical densities localized to cover elements of a particular type of covering – a so called (K_1, K_2)-*cover* – where the cover is over the region of turbulent activity. For simplicity, this region will be taken as a ball of radius R_0 centered at the origin and, to reflect the turbulence literature, is henceforth referred to as the *integral domain* (also known as the *macro-scale domain*). The time interval on which we localize is motivated by the physical theories of turbulence and assumed to satisfy,

$$T \geq \frac{R_0^2}{\nu}.$$

The (K_1, K_2)-covers are now defined.

Definition 1. *Let $K_1, K_2 \in \mathbb{N}$ and $0 \leq R \leq R_0$. The cover of the integral domain $B(0, R_0)$ by the n balls, $\{B(x_i, R)\}_{i=1}^n$ is a (K_1, K_2)-*cover at scale R *if*

$$\left(\frac{R_0}{R}\right)^3 \leq n \leq K_1 \left(\frac{R_0}{R}\right)^3$$

and, for any $x \in B(0, R_0)$, x is contained in at most K_2 balls from the cover.

In the hereafter, all covers are understood to be (K_1, K_2)- covers at scale R. The positive integers K_1 and K_2 represent the maximum allowed *global* and *local multiplicities*, respectively.

In order to localize a physical density to a cover element we incorporate certain *refined* cut-off functions. For the cover element around the point x_i, let $\phi_i(x, t) = \eta(t)\psi(x)$ where $\eta \in C^\infty(0, T)$ and $\psi \in C_0^\infty(B(x_i, 2R))$ satisfy

$$0 \leq \eta \leq 1, \qquad \eta = 0 \text{ on } (0, T/3), \qquad \eta = 1 \text{ on } (2T/3, T), \qquad \frac{|\partial_t \eta|}{\eta^\delta} \leq \frac{C_0}{T}, \tag{1}$$

and

$$0 \leq \psi \leq 1, \qquad \psi = 1 \text{ on } B(x_i, R), \qquad \frac{|\partial_i \psi|}{\psi^\rho} \leq \frac{C_0}{R}, \qquad \frac{|\partial_i \partial_j \psi|}{\psi^{2\rho - 1}} \leq \frac{C_0}{R^2}. \tag{2}$$

where $3/4 < \delta, \rho < 1$.

By ϕ_0 we denote the cut-off function associated with the integral domain – the ball centered at $x = 0$ of radius R_0 – satisfying the above properties.

Comparisons will be necessary between averaged quantities localized to cover elements at some scale R and averaged quantities at the integral scale. To accommodate this we impose several additional conditions for x_i near the boundary of $B(0, R_0)$. If $B(x_i, R) \subset B(0, R_0)$ we assume $\psi \leq \psi_0$. Alternatively, when $B(x_i, R) \not\subset B(0, R_0)$ we stipulate that $\psi = 1$ on $B(x_0, R) \cap B(0, R_0)$, satisfies (2), and we additionally have:

> $\psi = \psi_0$ on the intersection of $S(x_0, R_0, 2R_0)$ and the cone with apex at the origin and with boundaries passing through the intersection of the circle centered at the origin of radius R_0 and the boundary of $B(x_i, R)$,

and,

> $\psi = 0$ on the intersection of the three sets $B(0, R_0) \setminus B(x_i, 2R)$, $S(0, R_0, 2R_0)$, and the outside of the cone with apex at the origin and boundaries passing through the intersection of the circle centered at the origin of radius R_0 and the boundary of $B(x_i, 2R)$.

The above apparatus is employed to study properties of a physical density at a *physical scale* R associated with the integral domain $B(0, R_0)$ in a manner which we now illustrate. Let θ be a physical density (e.g. a flux density) and define its localized spatio-temporal average on a cover element at scale R around x_i as

$$\tilde{\Theta}_{x_i,R} = \frac{1}{T} \int_0^T \frac{1}{R^3} \int_{B(x_i,2R)} \theta(x,t) \phi_i^\delta(x,t) \, dx \, dt,$$

where $0 < \delta \leq 1$, and let $\langle \Theta \rangle_R$ denote the ensemble average over localized averages associated with cover elements,

$$\langle \Theta \rangle_R = \frac{1}{n} \sum_{i=1}^n \tilde{\Theta}_{x_i,R}.$$

Examining the values obtained by ensemble averaging the averages associated to a variety of covers at a fixed scale allows us to draw conclusions about the flux density θ at comparable and greater scales. For instance, stability (i.e. near constancy) of $\{\langle \Theta \rangle_R\}$ indicates that the sign of θ is essentially uniform at scales comparable to or greater than R. On the other hand, if the sign were not essentially uniform at scale R, particular covers could be arranged to enhance negative and positive regions and thus give a wide range of sign varying values in $\{\langle \Theta \rangle_R\}$. In order, then, to show the essential positivity of an *a priori* sign varying density θ at a scale R, it is sufficient to show the positivity of all elements of $\{\langle \Theta \rangle_R\}$.

Conversely, if θ is an *a priori* non-negative density, then ensemble averages are all comparable to the integral scale average across the range $0 < R \leq R_0$. We make this notion precise in the following lemma.

Lemma 2. *Let $f(x,t) \in L^1_{loc}((0,T) \times \mathbb{R}^3)$ be non-negative. Let $\{x_i\}_{i=1}^n$ be centers of elements of a (K_1, K_2)-cover of $B(x_0, R_0)$ at scale $R < R_0$. Setting*

$$F_0 = \frac{1}{T} \int_0^T \frac{1}{R_0^3} \int f(x,t) \phi_0(x,t) \, dx \, dt,$$

and

$$F_{x_i,R} = \frac{1}{T} \int_0^T \frac{1}{R^3} \int f(x,t) \phi_{x_i,R}(x,t) \, dx \, dt,$$

we have

$$\frac{1}{K_1} F_0 \leq \langle F \rangle_R \leq K_2 F_0. \tag{3}$$

Proof. Recalling that $\phi_{x_i,R} \leq \phi_0$ and the definition of (K_1, K_2)-covers we have that,

$$\begin{aligned}\langle F \rangle_R &= \frac{1}{T} \int_0^T \frac{1}{nR^3} \int f(x,t) \sum_{i=1}^n \phi_{x_i,R}(x,t)\, dx\, dt \\ &\leq \frac{1}{T} \int_0^T \frac{1}{R^3} \frac{R^3}{R_0^3} \int f(x,t) K_2 \phi_0(x,t)\, dx\, dt = K_2 F_0,\end{aligned}$$

and,

$$\begin{aligned}\langle F \rangle_R &= \frac{1}{T} \int_0^T \frac{1}{nR^3} \int f(x,t) \sum_{i=1}^n \phi_{x_i,R}(x,t)\, dx\, dt \\ &\geq \frac{1}{T} \int_0^T \frac{1}{R^3} \frac{1}{K_1} \frac{R^3}{R_0^3} \int f(x,t) \phi_0(x,t)\, dx\, dt = \frac{1}{K_1} F_0.\end{aligned}$$

□

For additional discussion of (K_1, K_2)-covers and ensemble averages, including some computational illustrations of the process, see [14].

3. 3D Incompressible MHD Equations.

Our mathematical setting is that of *weak solutions* to the 3D magnetohydrodynamic equations over $\mathbb{R}^3$ (cf. [23] for the foundational theory). Define $\mathcal{V} = \{f \in L^2(\mathbb{R}^3) : \nabla \cdot f = 0\}$ (where the divergence free condition is in the sense of distributions) and let V be the closure of $\mathcal{V}$ under the norm of the Sobolev space, $(H^1(\mathbb{R}^3))^3$, and, H, the closure of $\mathcal{V}$ under the L^2 norm. By a solution to 3D MHD we mean a weak (distributional) solution to the following coupled system (3D MHD):

$$\begin{aligned} u_t - \nu \triangle u + (u \cdot \nabla) u - (b \cdot \nabla) b + \nabla (p + |b|^2/2) = 0, \\ b_t - \eta \triangle b + (u \cdot \nabla) b - (b \cdot \nabla) u = 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x) \in V, \\ b(x,0) = b_0(x) \in V, \end{aligned}$$

where η and ν are the magnetic diffusivity and kinematic viscosity respectively and $p(x,t)$ is the fluid pressure.

Our present work utilizes *suitable weak solutions* for MHD, our definition of which follows that in [20].

Definition 3. *The functions u, b, p constitute a suitable weak solution to 3D MHD if they are a weak solution and further satisfy the following generalized energy inequality:*

$$\begin{aligned}
&\int_0^T \int (\nu|\nabla u(x,t)|^2 + \eta|\nabla b(x,t)|^2)\phi(x,t)\, dx\, dt \qquad (4)\\
&\leq \frac{1}{2}\int_0^T \int (|u(x,t)|^2 + |b(x,t)|^2)\phi_t(x,t)\, dx\, dt\\
&\quad + \frac{1}{2}\int_0^T \int (\nu|u(x,t)|^2 + \eta|b(x,t)|^2)\Delta\phi(x,t)\, dx\, dt\\
&\quad + \frac{1}{2}\int_0^T \int (|u(x,t)|^2 + |b(x,t)|^2 + 2p(x,t))(u(x,t)\cdot\nabla\phi(x,t))\, dx\, dt\\
&\quad - \int_0^T \int (u(x,t)\cdot b(x,t))(b(x,t)\cdot\nabla\phi(x,t))\, dx\, dt,
\end{aligned}$$

for a.e. $T \in (0,\infty)$ and any non-negative $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,\infty))$.

Existence of suitable weak solutions for MHD is proven in [20] using an adaptation of the traditional method for NSE first presented in [7]. We will make use of certain regularity properties which are evident from the existence theorem given in [20].

Proposition 4. *For u_0, $b_0 \in H$ and $u_0 \in W^{4/5,5/3}$, suppose (u,b,p) constitutes a suitable weak solution to 3D MHD. Then, (u,b,p) satisfies,*

$$u, b \in L^\infty(0,\infty; H),\, u, b \in L^3(0,\infty; L^3(\mathbb{R}^3)),$$
$$\nabla u, \nabla b \in L^2(0,\infty; L^2(\mathbb{R}^3)),\, p \in L^{3/2}(0,\infty; L^{3/2}(\mathbb{R}^3)).$$

The fact that suitable weak solutions only satisfy a generalized energy inequality (as opposed to equality) introduces the possibility that energy is dissipated not only by viscosity or resistivity but also by singularities. In the case that the weak solution in question is *regular*, equality is attained in the generalized energy inequality, (4), and the possibility of potential loss of flux due to singularities is eliminated. To streamline discussion we establish cascades for regular solutions and include a result for the non-regular case only in the context of the cascade for the modified (due to energy loss from possible singularities) total energy flux (c.f. Section 4). The study of the possible energy loss due to singularities is itself an interesting subject but the case of MHD is not sufficiently distinct from that of NSE (which can be found in [10]) to justify an independent exposition.

4. Total Energy Cascade.

The total energy flux through the boundary of the ball B, over the interval $(0,T)$, is given (cf. [3]) by,

$$\frac{1}{2}\int_0^T \int_{\partial B} (|u|^2 + |b|^2 + 2p)\, \hat{n}\cdot u\, dx\, dt - \int_0^T \int_{\partial B} (u\cdot b)(\hat{n}\cdot b)\, dx\, dt,$$

where $\hat{n}$ is the unit normal vector directed inward. Our analytic results are enabled by substituting the inwardly directed vector field $\nabla\phi$ for $\hat{n}$ where ϕ is a refined cut-off function for the ball $B(x_0,R)$. Our localized, space-time averaged total energy flux into the ball centered at the spatial point x_0 of radius R, over the interval $(0,T)$, is then defined as,

$$F^E_{x_0,R} := \frac{1}{2}\int_0^T \int (|u|^2 + |b|^2 + 2p)(u\cdot\nabla\phi)\, dx\, dt - \int_0^T \int (u\cdot b)(b\cdot\nabla\phi)\, dx\, dt.$$

The following remark on the genesis of the last term, the advection of the cross-helicity via the magnetic field, is informative. The transfer of magnetic to kinetic energy is driven by the Lorentz force while the stretching of the magnetic field lines is responsible for the transfer of kinetic energy to magnetic energy. Since these are complimentary, the sum cancels in the non-localized case. Locally, however, we obtain a flux-type term,

$$\int_0^T \int ((b\cdot\nabla)b\cdot\phi u + (b\cdot\nabla)u\cdot\phi b)\,dx\,dt = -\int_0^T\int (u\cdot b)(b\cdot\nabla\phi)\,dx\,dt. \tag{5}$$

The appearance of this term is interesting because it is only non-zero if there is some degree of non-locality in the energy transfer *between the two fields*.

In the following we work in a fixed integral domain, $B(x_0, R_0)$, with associated cut-off $\phi_0 = \phi_{x_0,R_0}$. Certain integral domain quantities will be used to determine lower bounds on the inertial ranges over which our cascades are rigorously shown to persist. Our labeling scheme follows. The integral scale space-time averaged kinetic and magnetic energies are defined in terms of a technical parameter, δ, as,

$$e_0^u = e_0^u(\delta) = \frac{1}{T}\int_0^T \frac{1}{R_0^3}\int \frac{1}{2}|u|^2\phi_0^\delta\,dx\,dt \quad\text{and}\quad e_0^b = e_0^b(\delta) = \frac{1}{T}\int_0^T \frac{1}{R_0^3}\int \frac{1}{2}|b|^2\phi_0^\delta\,dx\,dt,$$

and the integral scale time-space averaged enstrophies by,

$$E_0^u = \nu\frac{1}{T}\int_0^T \frac{1}{R_0^3}\int |\nabla u|^2\phi_0\,dx\,dt \quad\text{and}\quad E_0^b = \eta\frac{1}{T}\int_0^T \frac{1}{R_0^3}\int |\nabla b|^2\phi_0\,dx\,dt.$$

Regarding notation, we indicate the combined kinetic and magnetic energies or enstrophies by omitting the superscript (i.e. $e_0 := e_0^u + e_0^b$) and, because e_0 is decreasing with δ, take liberties suppressing the dependence of e_0 on δ with the understanding that the indicated quantity is that associated with the smallest appropriate value for δ.

We will establish that the cascade persists over a range of scales bound above by the integral scale and below by a modified Taylor micro-scale. The Taylor micro-scale is (c.f. [3]),

$$\tau = \left(\frac{\nu e_0}{E_0}\right)^{1/2}.$$

The modification will depend in part on the *magnetic Prandtl number*, denoted by Pr, a non-dimensional number given by the ratio of kinematic viscosity to magnetic diffusivity, i.e.,

$$Pr = \frac{\nu}{\eta}.$$

Essentially its roll in the modification is to incorporate information about the magnetic diffusivity which is absent from our prescribed time scale (recall $T > R_0^2/\nu$) and the Taylor micro-scale.

Theorem 5. *Assume u and b are suitable weak solutions to 3D MHD with sufficient regularity that equality holds in* (4). *Let $\{x_i\}_{i=1}^n \subset B(x_0, R_0)$ be the centers of a (K_1, K_2)-cover at scale R where the cut-off functions are defined with $T \geq R^2/\nu$. For a scale and cover independent positive parameter*

$$\beta := \left(\frac{1}{2CK_1K_2(1+Pr^{-1})}\right)^{1/2},$$

where C is a constant determined by structural properties of 3D MHD and our cut-off functions, if

$$\tau < \beta R_0,$$

then,

$$\frac{1}{2K_1}E_0 \leq \langle F^E\rangle_R \leq 2K_2E_0,$$

provided R is contained in the interval $[\tau/\beta, R_0]$.

Proof. Assuming the premises above and in virtue of (4), we have for any cover element that,

$$F^E_{x_i,R} \geq \int_0^T \int (\nu|\nabla u|^2 + \eta|\nabla b|^2)\phi_{x_i,R}\, dx\, dt$$
$$- \left| \frac{1}{2}\int_0^T \int (|u|^2 + |b|^2)\partial_t \phi_{x_i,R}\, dx\, dt + \frac{1}{2}\int_0^T \int (\nu|u|^2 + \eta|b|^2)\Delta\phi_{x_i,R}\, dx\, dt \right|.$$

Recalling the property of our cut-off function,

$$|\partial_t \phi_{x_i,R}| \leq c_0 \frac{\phi^\rho_{x_i,R}}{T},$$

as well as the fact,

$$\frac{1}{T} \leq \frac{\nu}{R^2} = Pr\frac{\eta}{R^2},$$

we conclude that,

$$\frac{1}{2}\int_0^T \int (|u|^2 + |b|^2)\partial_t \phi_{x_i,R}\, dx\, dt \leq c_0\frac{\nu}{R^2}\int_0^T \int |u|^2 \phi^\rho_{x_i,R}\, dx\, dt + c_0 Pr\frac{\eta}{R^2}\int_0^T \int |b|^2\phi^\rho_{x_i,R}\, dx\, dt.$$

Our cut-off functions also satisfied,

$$|\Delta\phi_{x_i,R}| \leq c_0\frac{\phi^{2\rho-1}}{R^2},$$

and, consequently,

$$\frac{1}{2}\int_0^T \int (\nu|u|^2 + \eta|b|^2)\Delta\phi_{x_i,R}\, dx\, dt \leq c_0\frac{\nu}{R^2}\int_0^T \int |u|^2\phi^{2\rho-1}_{x_i,R}\, dx\, dt + c_0\frac{\eta}{R^2}\int_0^T \int |b|^2\phi^{2\rho-1}_{x_i,R}\, dx\, dt.$$

Noting that $\rho > 2\rho - 1$ we have, upon combining the above estimates, that,

$$F^E_{x_i,R} \geq \int_0^T \int (\nu|\nabla u|^2 + \eta|\nabla b|^2)\phi_{x_i,R}\, dx\, dt - \frac{c_0}{R^2}\int_0^T \int (\nu|u|^2 + \eta(1+Pr)|b|^2)\phi^{2\rho-1}_{x_i,R}\, dx\, dt.$$

Using Lemma 2, we observe that,

$$K_2 E_0 \geq \left\langle \frac{1}{T}\int_0^T \frac{1}{R^3}\int (\nu|\nabla u|^2 + \eta|\nabla b|^2)\phi_{x_i,R}\, dx\, dt \right\rangle_R \geq \frac{1}{K_1}E_0,$$

and,

$$\begin{aligned} \left\langle \frac{1}{T}\int_0^T \frac{1}{R^3}\int \frac{1}{2}(\nu|u|^2 + \eta(1+Pr)|b|^2)\phi^{2\rho-1}_{x_i,R}\, dx\, dt \right\rangle_R &\leq \nu K_2 e^u_0 + \eta(1+Pr)K_2 e^b_0 \\ &\leq \nu K_2\big(1 + Pr^{-1}\big)e_0. \end{aligned}$$

We can thus interpolate the ensemble average between integral scale quantities as,

$$\frac{1}{K_1}E_0 - \nu\frac{c_0 K_2(1+Pr^{-1})}{R^2}e_0 \leq \langle F^E \rangle_R \leq K_2 E_0 + \nu\frac{c_0 K_2(1+Pr^{-1})}{R^2}e_0.$$

Note that the upper bound follows in virtue of the assumed regularity (i.e. equality in (4)) and is the only place where this assumption is used. In particular, the lower bound holds for non-regular solutions.

Continuing, we now specify a value for β, the modification to the inertial range, to be,

$$\beta = \left(\frac{1}{2c_0 K_1 K_2(1+Pr^{-1})}\right)^{1/2}.$$

Using the inertial range condition that $\tau/\beta \leq R \leq R_0$, we have,

$$\nu \frac{c_0 K_2 (1 + Pr^{-1})}{R^2} e_0 \leq \frac{1}{2K_1} E_0.$$

The final lower bound for the ensemble average then becomes,

$$\left\langle F^E \right\rangle_R \geq \frac{1}{2K_1} E_0.$$

The upper bound follows trivially with our definition of β and we conclude that,

$$\frac{1}{2K_1} E_0 \leq \left\langle F^E \right\rangle_R \leq 2K_2 E_0.$$

□

Remark 6. *The dependence of the length of the inertial range on Pr has consequences for when the above result is most physically relevant. The correction to the Taylor micro-scale is minimized when $Pr^{-1} \lesssim 1$, that is, when $\eta \lesssim \nu$. This is exactly the scenario for turbulence of plasmas in astronomical settings such as the solar wind and the interstellar medium.*

We briefly discuss how to obtain a result for possibly non-regular suitable weak solutions. As noted in the proof, the assumption of regularity was only used in establishing the upper bound on the ensemble average. This is the issue we now address. The physical cause of a strict inequality in the generalized energy inequality is interpreted as the loss of energy due to possible singularities. This will be denoted by F_ϕ^∞ (or $F_{x_i,R}^\infty$ if ϕ is the cut-off for a (K_1, K_2)-cover element) and is defined as the value that fills in the inequality (4),

$$\int_0^T \int (\nu|\nabla u|^2 + \eta|\nabla b|^2)\phi \, dx \, dt = \frac{1}{2}\int_0^T \int (|u|^2 + |b|^2)\phi_t \, dx \, dt + \frac{1}{2}\int_0^T \int (\nu|u|^2 + \eta|b|^2)\Delta\phi \, dx \, dt \tag{6}$$

$$+ \frac{1}{2}\int_0^T \int (|u|^2 + 2|b|^2 + 2p)(u \cdot \nabla\phi) \, dx \, dt - \int_0^T \int (u \cdot b)(b \cdot \nabla\phi) \, dx \, dt - F_\phi^\infty.$$

To account for the possible strictness of the above inequality we replace the fluxes considered previously with fluxes modified to include the possible loss of energy due to singularities. For example, in the case of the total energy flux, we establish an interpolative bound on ensemble averages corresponding to the localized *modified total energy fluxes*,

$$F_\phi^{E,\infty} = F_\phi^E - F_\phi^\infty.$$

Positivity and near-constancy results are then given in terms of the modified flux specified above. The *cascade of total energy modified due to possible non-regularity* is then given by the following theorem, the proof of which is identical modulo a substitution of $F_\phi^{E,\infty}$ for F_ϕ^E to the proof of (5).

Theorem 7. *Assume u and b are suitable weak solutions to 3D MHD. Let $\{x_i\}_{i=1}^n \subset B(x_0, R_0)$ be the centers of a (K_1, K_2)-cover at scale R where the cut-off functions are defined with $T \geq R^2/\nu$. For a scale- and cover-independent, positive parameter,*

$$\beta := \left(\frac{1}{2CK_1K_2(1 + Pr^{-1})} \right)^{1/2},$$

where C is a constant determined by structural properties of 3D MHD and our cut-off functions, if

$$\tau < \beta R_0,$$

then,

$$\frac{1}{2K_1} E_0 \leq \left\langle F^{E,\infty} \right\rangle_R \leq 2K_2 E_0,$$

provided R is contained in the interval $[\tau/\beta, R_0]$.

5. Total Fluid Energy Cascade.

Our attention is now turned to establishing conditions under which a cascade of the total fluid energy will exist. The localized flux quantity of interest is,

$$\frac{1}{2}\int_0^T\int (|u|^2+2p)(u\cdot\nabla\phi)\,dx\,dt = -\int_0^T\int ((u\cdot\nabla)u+p)\cdot(\phi\,u)\,dx\,dt.$$

Results are ultimately intended for the dimensional form of the equations (as was the case for the cascade of total energy). Our estimates, however, will substantially benefit from switching to the *dimensionless form* of the equations. Some general remarks illustrating this process are now included. Starting with a dimensional problem where the integral domain has radius R_0 (the characteristic length scale) and ν is the kinematic viscosity (these will be our fundamental dimensions), we will establish the existence of a dimensionless cascade over an inertial range given by,

$$\left(\frac{R_0}{R}\right)^{\delta}\frac{e_0^*}{E_0^*} < \beta,$$

where the dimensionless analogues to the dimensional solutions and variables are indicated with asterisks and the analogous normalized integral domain quantities are,

$$e_0^* = \int_0^1\int \frac{1}{2}(|u^*|^2+|b^*|^2)\phi_0\,dx^*dt^*,$$
$$E_0^* = \int_0^1\int \frac{1}{2}(|\nabla u^*|^2+|\nabla b^*|^2)\phi_0\,dx^*dt^*,$$

and β is a dimensionless constant.

This has consequences for the dimensional setting. In particular, we have the quantitative relationship,

$$\frac{e_0}{E_0} = R_0^2\frac{e_0^*}{E_0^*},$$

which follows from a change of variable and the chain rule. Furthermore, setting our characteristic time scale to be $T = R_0^2\nu^{-1}$, we see the integral scale kinetic energy relates to its dimensionless analogue in the following way,

$$\begin{aligned}
e_0^u = \frac{1}{TR_0^3}\int_0^T\int |u|^2\phi_0\,dx\,dt &= \frac{1}{TR_0^3}\int_0^T\int \frac{R_0^2}{T^2}\left|\frac{u(x,t)}{u_0}\right|^2\phi_0(x,t)\,dx\,dt\\
&= \frac{1}{TR_0^3}\int_0^T\int \frac{R_0^2}{T^2}|u^*(x^*,t^*)|^2\phi_0(x^*,t^*)\;R_0^3\,dx^*\;T\,dt^*\\
&= \frac{R_0^2}{T^2}\int_0^T\int |u^*(x^*,t^*)|^2\phi_0(x^*,t^*)\,dx^*\,dt^*\\
&= \frac{R_0^2}{T^2}\,e_0^{u^*}.
\end{aligned}$$

Similarly,

$$e_0^b = \frac{R_0^2}{T^2}e_0^{b^*},\quad E_0^u = \frac{1}{T^2}E_0^{u^*},\quad \text{and,}\quad E_0^b = \frac{1}{T^2}E_0^{b^*}.$$

For the flux presently of interest, total (fluid) energy, setting,

$$F_{x_i,R} = \frac{1}{TR^3}\int_0^T \int ((u\cdot\nabla)u + \nabla p)\cdot(\phi_{x_i,R}u)\,dx\,dt,$$
$$F^*_{x_i,R} = \frac{R_0^3}{R^3}\int_0^1 \int ((u^*(x^*,t^*)\cdot\nabla_*)u^*(x^*,t^*) + \nabla_* p^*(x^*,t^*))\cdot\phi_{x_i,R}(x^*,t^*)u^*(x^*,t^*)\,dx^*\,dt^*,$$

we have, upon transforming the former into the latter, that

$$F_{x_i,R} = \frac{\nu}{T^2}F^*_{x_i,R}.$$

Assuming now the existence of the cascade for the dimensionless system we can recover a cascade for the dimensional scenario as we now illustrate. We will see theorems pertaining to dimensionless cascades which assert that, for certain dimensionless quantities β_u and β_b, if,

$$\frac{R_0}{\beta_u}\left(\frac{e_0^{u^*}}{E_0^{u^*}}\right)^{1/4} < R < R_0 \quad \text{and} \quad \frac{R_0}{\beta_b}\left(\frac{e_0^{b^*}}{E_0^{b^*}}\right)^{1/4} < R < R_0,$$

then,

$$\frac{1}{2K_*}E_0^* \le \langle F^*\rangle_R \le 2K_* E_0^*.$$

By the quantitative relations identified above, the consequence for the dimensional scenario is, if,

$$\frac{R_0^{1/2}}{\beta_u}\left(\frac{e_0^{u}}{E_0^{u}}\right)^{1/4} < R < R_0 \quad \text{and} \quad \frac{R_0^{1/2}}{\beta_b}\left(\frac{e_0^{b}}{E_0^{b}}\right)^{1/4} < R < R_0,$$

then,

$$\frac{\nu}{2K_*}E_0 \le \langle F\rangle_R \le 2\nu K_* E_0.$$

One last comment regarding the interplay between dimensional and dimensionless structures is that, upon making a change of variables with $x^* = x/R_0$ and $t^* = t/T$ and applying the chain rule, our refined cut-off functions are seen to enjoy the following bounds,

$$|\nabla_*\phi(x^*,t^*)| \le \frac{R_0}{R}c_0|\phi(x^*,t^*)|^\rho \quad \text{and} \quad |\phi_{t^*}(x^*,t^*)| \le c_0|\phi^\rho(x^*,t^*)|. \tag{7}$$

The above discussion justifies a switch to the dimensionless setting which we presently make. We subsequently suppress the asterisks used above to indicate non-dimensionality noting that for the remainder of this section all functions and variables are dimensionless.

Following [23], dimensionless 3D MHD is,

$$\partial_t u - \frac{1}{Re}\Delta u = -(u\cdot\nabla)u - \nabla p - S\left(\nabla\frac{|b|^2}{2} - (b\cdot\nabla)b\right),$$
$$\partial_t b - \frac{1}{Rm}\Delta b = \nabla\times(b\times u),$$
$$\nabla\cdot u = \nabla\cdot b = 0,$$
$$u(x,0) = u_0(x) \in L^2(\mathbb{R}^3),$$
$$b(x,0) = b_0(x) \in L^2(\mathbb{R}^3),$$

where Re and Rm are the Reynolds and magnetic Reynolds numbers respectively and the non-dimensional number, S, is defined in terms of the Hartmann number, M, to be $S = M^2/(ReRm)$.

In order to obtain a local cancellation between coupled non-linear terms, we assume sufficient regularity so that the following derivation is justified. Taking the scalar product of the equation

of motion by ϕu and the induction equation by $S\phi b$, integrating, and rearranging we obtain a flux density for total (fluid) energy into the ball $B(x_i/R_0, R/R_0)$ with associated cut-off function ϕ,

$$\begin{aligned}\int_0^1\int\left(\frac{1}{2}|u|^2+p\right)\cdot(\nabla\phi\cdot u)\,dx\,dt &= \frac{1}{Re}\int_0^1\int|\nabla u|^2\phi\,dx\,dt+\frac{S}{Rm}\int_0^1\int|\nabla b|^2\phi\,dx\,dt\\ &\quad - S\int_0^1\int\nabla\phi\cdot(b\times(u\times b))\,dx\,dt\\ &\quad -\frac{1}{2}\int_0^1\int|u|^2\phi_t\,dx\,dt-\frac{1}{2Re}\int_0^1\int|u|^2\Delta\phi\,dx\,dt\\ &\quad -\frac{S}{2}\int_0^1\int|b|^2\phi_t\,dx\,dt-\frac{S}{2Rm}\int_0^1\int|b|^2\Delta\phi\,dx\,dt.\end{aligned}$$

Bounds for the lower order terms on the right hand side follow. The last four terms are bounded in a manner similar to that seen in the proof of (5). Here, however, we acknowledge the change of variable and cite the properties of our refined cut-off functions (c.f. (7)) as well as the chain rule to obtain,

$$\begin{aligned}\frac{1}{2}\int_0^1\int|u|^2\phi_t\,dx\,dt &\le c_0 Re\frac{1}{Re}\left(\frac{R_0}{R}\right)^2\int_0^1\int|u|^2\phi^{4\rho-3}\,dx\,dt,\\ \frac{1}{2Re}\int_0^1\int|u|^2\Delta\phi\,dx\,dt &\le c_0\frac{1}{Re}\left(\frac{R_0}{R}\right)^2\int_0^1\int|u|^2\phi^{4\rho-3}\,dx\,dt,\\ \frac{S}{2}\int_0^1\int|b|^2\phi_t\,dx\,dt &\le c_0 Rm\frac{S}{Rm}\left(\frac{R_0}{R}\right)^2\int_0^1\int|b|^2\phi^{\rho}\,dx\,dt,\\ \frac{S}{2Rm}\int_0^1\int|b|^2\Delta\phi\,dx\,dt &\le c_0\frac{S}{Rm}\left(\frac{R_0}{R}\right)^2\int_0^1\int|b|^2\phi^{4\rho-3}\,dx\,dt.\end{aligned}\tag{8}$$

Repeatedly using Hölder's inequality, the Gagliardo-Nirenberg inequality, and Young's inequality, the following bound on the remaining term is evident,

$$\begin{aligned}c_0S\frac{R_0}{R}\int_0^1\int\phi^{\rho}(b\times(u\times b))\,dx\,dt &\le c_0S\frac{R_0}{R}\int_0^1\int(|u|^{1/2}\phi^{\rho-3/4})(|b|^{1/2})(|u|^{1/2}|b|^{3/2}\phi^{3/4})\,dx\,dt\\ &\le c_0S\frac{R_0}{R}\int_0^1||u\phi^{2\rho-3/2}||_2^{1/2}||b||_2^{1/2}||(|u|^{1/2}|b|^{3/2}\phi^{3/4})||_2\,dt\\ &\le c_0\frac{R_0}{R}\left(\frac{M^{1/2}Rm^{1/2}}{Re^{1/4}}\right)\left(\frac{M^{3/2}}{Re^{3/4}Rm^{3/2}}\right)\\ &\quad\cdot\int_0^1||u\phi^{2\rho-3/2}||_2^{1/2}||b||_2^{1/2}||u||_2^{1/2}||\nabla(b\phi^{1/2})||_2^{3/2}\,dt\\ &\le c_0\frac{M^2Rm^2R_0^4}{ReR^4}\big(\sup_t||u||_2^2\big)\big(\sup_t||b||_2^2\big)\int_0^1||u\phi^{4\rho-3}||_2^2\,dt\\ &\quad+\frac{S}{4Rm}\int_0^1||\nabla(b\phi^{1/2})||_2^2\,dt\\ &\le c_0(MRm)^2\big(\sup_t||u||_2^2\sup_t||b||_2^2\big)\frac{1}{Re}\frac{R_0^4}{R^4}\int_0^1||u\phi^{4\rho-3}||_2^2\,dt\\ &\quad+\frac{S}{4Rm}\int_0^1||(\nabla b)\phi^{1/2}||_2^2\,dt+\frac{c_0S}{Rm}\frac{R_0^4}{R^4}\int_0^1||b\phi^{4\rho-3}||_2^2\,dt.\end{aligned}\tag{9}$$

Taking ensemble averages and passing where appropriate (via Lemma 2) to the quantities e_0^u, e_0^b, E_0^u, and E_0^b, we have ,

$$\begin{aligned}\langle F\rangle_R &\geq \frac{1}{K_1}\frac{1}{Re}E_0^u - c_0\frac{K_2}{Re}\big(1+Re+(MRm)^2\big(\sup_t \|u\|_2^2 \sup_t \|b\|_2^2\big)\big)\frac{R_0^4}{R^4}e_0^u \\ &\quad + \frac{1}{K_1}\frac{S}{Rm}E_0^b - c_0\frac{K_2 S}{Rm}\big(2+Rm\big)\frac{R_0^4}{R^4}e_0^b \\ &= \frac{1}{K_1}\frac{1}{Re}E_0^u - C_u\frac{K_2}{Re}\frac{R_0^4}{R^4}e_0^u + \frac{1}{K_1}\frac{S}{Rm}E_0^b - C_b\frac{K_2 S}{Rm}\frac{R_0^4}{R^4}e_0^b,\end{aligned}$$

where in the last line we have set,

$$C_u = c_0\bigg(1+Re+(MRm)^2\big(\sup_t \|u\|_2^2 \sup_t \|b\|_2^2\big)\bigg) \text{ and } C_b = c_0\big(2+Rm\big).$$

Similarly, an upper bound is,

$$\langle F\rangle_R \leq K_2\frac{1}{Re}E_0^u + C_u K_2\frac{1}{Re}\frac{R_0^4}{R^4}e_0^u + K_2\frac{S}{Rm}E_0^b + C_b K_2\frac{S}{Rm}\frac{R_0^4}{R^4}e_0^b.$$

Defining now the parameters for a correction to the inertial range by,

$$\beta_u = \left(\frac{1}{2K_1K_2C_u}\right)^{1/4} \text{ and } \beta_b = \left(\frac{1}{2K_1K_2C_b}\right)^{1/4},$$

we have justified the following theorem.

Theorem 8. *Let $\{x_i\}_{i=1}^n \subset B(x_0, R_0)$ be the centers of a (K_1, K_2)-cover at scale R. For β_u and β_b defined above, if*

$$\tau := \max\left\{\left(\frac{e_0^u}{E_0^u}\right)^{1/4}, \left(\frac{e_0^b}{E_0^b}\right)^{1/4}\right\} < \min\{\beta_u, \beta_b\} =: \beta,$$

then for scales R where $\tau/\beta \leq R \leq R_0$, we have,

$$\frac{1}{2K_1}\left(\frac{1}{Re}E_0^u + \frac{S}{Rm}E_0^b\right) \leq \langle F\rangle_R \leq 2K_2\left(\frac{1}{Re}E_0^u + \frac{S}{Rm}E_0^b\right).$$

Remark 9. *It is worth noting that the condition triggering the cascade is essentially a requirement that the* gradients *of the velocity and the magnetic fields are large (averaged, over the integral domain) with respect to the fields themselves; this will hold in the regions of high* spatial complexity *of the flow (the correction parameter β depends on certain* a priori *bounded quantities; however, none of them involve gradients).*

Remark 10. *A particular scenario where the above result is pertinent is one where $S/Rm \simeq 1$ and the integral scale magnetic energy dominates the integral scale kinetic energy. The effect of the above assumption is to free the parameter β_u from its dependence on Re by substituting for the bound* (8)*, the bound*

$$\frac{1}{2}\int_0^1 \int |u|^2 \partial_t \phi_{x_i,R}\, dx\, dt \leq K_2 e_0^b.$$

This allows for very large values of Re and is applicable to settings involving the confinement of a liquid metal by a strong magnetic guide field [24]*. Without these assumptions on the flow the physical relevance of the above result is diminished by the fact that the dependences of β_u and β_b on the fluid and magnetic Reynold's numbers result in an inertial range which decreases in length as Rm and Re increase.*

An Improvement via Dynamic Alignment. Interestingly, the possible detraction from the positivity of the ensemble average by the remaining flux terms from the generalized energy inequality,

$$N_{x_i,R} := \int_0^1 \int |b|^2 (u \cdot \nabla \phi_{x_i,R})\, dx\, dt - \int_0^1 \int (u \cdot b)(b \cdot \nabla \phi_{x_i,R})\, dx\, dt,$$

is (partially) balanced by an observationally and numerically supported geometric depletion in the form of *scale dependent dynamic alignment* (c.f. in particular [4] as well as [3, 22, 21, 2]). Essentially, a decreasing length scale corresponds to an increase in the alignment or anti-alignment of the angle between u and b. It is a matter of vector calculus to rewrite $N_{x_i,R}$ as,

$$N_{x_i,R} = \int_0^1 \int \nabla \phi_{x_i,R} \cdot (b \times (u \times b))\, dx\, dt.$$

This allows the geometric depletion to be exploited since, as the alignment between u and b increases, the value $|u \times b|$ is diminished in a scale-dependent manner. Letting $\theta(u,b)$ denotes the angle between u and b our interpretation of this phenomenon is the implication,

$$|u| > \left(\frac{R}{R_0}\right) \Rightarrow \sin(\theta(u,b)) < \left(\frac{R}{R_0}\right)^{1/4}. \tag{10}$$

Theorem 11. *Let $\{x_i\}_{i=1}^n \subset B(x_0, R_0)$ be the centers of a (K_1, K_2)-cover at scale R. Assume the dynamical alignment assumption,* (10)*, holds. For appropriate values β_u and β_b, if*

$$\tau := \max\left\{ \left(\frac{e_0^u}{E_0^u}\right)^{1/3}, \left(\frac{e_0^b}{E_0^b}\right)^{1/3} \right\} < \min\{\beta_u, \beta_b\} =: \beta,$$

then, for scales R where $\tau/\beta \le R \le R_0$, we have,

$$\frac{1}{2K_1}\left(\frac{1}{Re} E_0^u + \frac{S}{Rm} E_0^b\right) \le \langle F \rangle_R \le 2K_2 \left(\frac{1}{Re} E_0^u + \frac{S}{Rm} E_0^b\right).$$

Proof. The only procedural difference between the present situation and that seen in (8) when bounding the quantity,

$$\int_0^1 \int (\nabla \phi) \cdot b \times (u \times b)\, dx\, dt.$$

To accommodate the dynamical alignment assumption we split the spatial integral over regions depending on the relationship $|u| < R/R_0$. If this is satisfied then,

$$\int_0^1 \int (\nabla \phi) \cdot b \times (u \times b)\, dx\, dt \le c_0 \left(\frac{R}{R_0}\right)^2 \int_0^1 \int |b|^2 \phi^{2\rho - 1}\, dx\, dt.$$

Otherwise we proceed as in (9) but with a minor modification to the normalized radius:

$$\begin{aligned}
\frac{R_0}{R} \int_0^1 \int \phi^\rho (b \times (u \times b))\, dx\, dt &\le \left(\frac{R_0}{R}\right)^{3/4} \int_0^1 \int (|u|^{1/2} \phi^{\rho - 3/4})(|b|^{1/2})(|u|^{1/2} |b|^{3/2} \phi^{3/4})\, dx\, dt \\
&\le c_0 (M Rm)^2 \big(\sup_t ||u||_2^2 \sup_t ||b||_2^2 \big) \frac{1}{Re} \left(\frac{R_0}{R}\right)^3 \int_0^1 ||u \phi^{4\rho - 3}||_2^2\, dt \\
&\quad + \frac{S}{4Rm} \int_0^1 ||(\nabla b) \phi^{1/2}||_2^2\, dt + \frac{c_0 S}{Rm} \left(\frac{R_0}{R}\right)^3 \int_0^1 ||b \phi^{4\rho - 3}||_2^2\, dt.
\end{aligned}$$

From here one proceeds as in the proof of Theorem 8 to establish β_u and β_b. The details are omitted to avoid redundancy. □

6. Individual Cascades.

We presently identify conditions under which direct cascades of several individual energy exchange mechanisms – including the exchange of energy between the velocity and magnetic fields – are evident in the context of the dimensionless equations. Throughout this section all solutions are assumed to be suitable weak solutions to the dimensionless 3D MHD system, regular enough for the localized energy equality to hold.

The localized (by function ϕ to a ball of radius R/R_0) flux term responsible for the kinetic to kinetic energy exchange driven by the advection of the velocity field is,

$$F^u_{x_i,R} := -\int_0^1 \int (u\cdot\nabla)u\cdot(\phi\, u)\,dx\,dt = \frac{1}{2}\int_0^1\int |u|^2(u\cdot\nabla\phi)\,dx\,dt. \tag{11}$$

Similarly, the (localized) magnetic to magnetic energy transfer driven by the advection of the velocity field is,

$$F^b_{x_i,R} := -\int_0^1 \int (u\cdot\nabla b)\cdot(\phi\, b)\,dx\,dt = \frac{1}{2}\int_0^1\int |b|^2(u\cdot\nabla\phi)\,dx\,dt. \tag{12}$$

The fluid pressure flux-type term is,

$$F^p_{x_i,R} := -\int_0^1 \int \nabla p\cdot(\phi\, u)\,dx\,dt = \int_0^1\int p\,(u\cdot\nabla\phi)\,dx\,dt. \tag{13}$$

As already mentioned, the transfer of magnetic to kinetic energy is driven by the Lorentz force while the stretching of the magnetic field lines is responsible for the transfer of kinetic energy to magnetic energy; combined (locally), they yield the following term – the advection of cross-helicity by the magnetic field,

$$F^{ub}_{x_i,R} := \int_0^1 \int ((b\cdot\nabla)b\cdot(\phi\, u) + (b\cdot\nabla)u\cdot(\phi\, b))\,dx\,dt = -\int_0^1\int (u\cdot b)(b\cdot\nabla\phi)\,dx\,dt. \tag{14}$$

Until now we have only investigated cascades associated with collections of flux-type terms including the term for the flux of the fluid pressure. Due to the unique structure of this term, additional effort is required to establish cascades of combined fluxes excluding the pressure flux. In particular, we will need to bound the quantity,

$$\int_0^1\int p\,(u\cdot\nabla\phi)\,dx\,dt.$$

For this, we'll need the following estimate,

$$\begin{aligned}
||u\phi^{1/2}||_3 &= \left(\int (|u|^{3/2}\phi^{3/4})(|u|^{3/2}\phi^{3/4})\,dx\right)^{1/3}\\
&\le \left(\left(\int |u|^2\phi\,dx\right)^{3/4}\left(\int |u|^6\phi^3\,dx\right)^{1/4}\right)^{1/3}\\
&\le C||u\phi^{1/2}||_2^{1/2}||\nabla(u\phi^{1/2})||_2^{1/2}.
\end{aligned}$$

Proceeding to the quantity of interest and employing the above bound, we have,

$$\begin{aligned}
\int_0^1 \int p\,(u\cdot\nabla\phi)\,dx\,dt &\leq C\frac{R_0}{R}\int_0^1 ||p\phi^{\rho-1/2}||_{3/2}||u\phi^{1/2}||_3\,dt \\
&\leq C\frac{R_0}{R}\int_0^1 ||p\phi^{\rho-1/2}||_{3/2}||u\phi^{1/2}||_2^{1/2}||\nabla(u\phi^{1/2})||_2^{1/2}\,dt \\
&\leq C\frac{R_0}{R}\Big(\int_0^1 ||p\phi^{\rho-1/2}||_{3/2}^{3/2}\,dt\Big)^{2/3}\Big(\int_0^1 ||u\phi^{1/2}||_2^6\,dt\Big)^{1/12}\Big(\int_0^1 ||\nabla(u\phi^{1/2})||_2^2\,dt\Big)^{1/4} \\
&\leq CRe^{1/3}\Big(\frac{R_0}{R}\Big)^{4/3}\Big(\int_0^T ||p\phi^{\rho-1/2}||_{3/2}^{3/2}\,dt\Big)^{8/9}\Big(\int_0^1 ||u\phi^{1/2}||_2^6\,dt\Big)^{1/9} \\
&\quad + \frac{1}{8Re}\int_0^1 ||\nabla(u\phi^{1/2})||_2^2\,dt \\
&\leq C_pRe^{1/3}\Big(\frac{R_0}{R}\Big)^{4/3}\Big(\int_0^1 ||u\phi^{1/2}||_2^2\,dt\Big)^{1/9} + C\frac{1}{Re}\Big(\frac{R_0}{R}\Big)^2\int_0^1 ||u\phi^{2\rho-1}||_2^2\,dt \\
&\quad + \frac{1}{4Re}\int_0^1 ||(\nabla u)\phi^{1/2}||_2^2\,dt, \qquad (15)
\end{aligned}$$

where we have used Hölder's inequality, the Gagliardo-Nirenberg inequality, and Young's inequality. Note that the quantity appearing above,

$$C_p = \big(\sup_t ||u\phi_0^{1/2}||_2\big)^{4/9}\Big(\int_0^1 ||p\phi_0^{\rho-1/2}||_{3/2}^{3/2}\,dt\Big)^{8/9}, \qquad (16)$$

is dimensionless and *a priori* bounded in virtue of regularity properties of suitable weak solutions (c.f. [20]). In addition, it contains no gradients and is independent of the particular cover element at scale R.

Regarding the first term on the right hand side of (15), in order to pass from ensemble averages of localized quantities to an integral scale quantity we will need a simple consequence of the finite form of Jensen's inequality. Specifically, for $\{a_i\}_{i=1}^n$ a set of non-negative values,

$$\sum_{i=1}^n \frac{a_i^{1/9}}{n} \leq \Big(\sum_{i=1}^n \frac{a_i}{n}\Big)^{1/9}.$$

Taking an ensemble average of normalized quantities yields,

$$\begin{aligned}
\frac{C_p}{n}\sum_{i=1}^n\Big(\frac{R_0}{R}\Big)^3\Big(\frac{R_0}{R}\Big)^{\frac{4}{3}}\Big(\int_0^1\int |u|^2\phi_{x_i,R}\,dt\Big)^{\frac{1}{9}} &\leq C_p\Big(\frac{R_0}{R}\Big)^4\Big(\frac{1}{n}\sum_{i=1}^n\int_0^1\Big(\frac{R_0}{R}\Big)^3\int |u|^2\phi_{x_i,R}\,dx\,dt\Big)^{\frac{1}{9}} \\
&\leq C_p\Big(\frac{R_0}{R}\Big)^4\big(K_2e_0^u\big)^{\frac{1}{9}}. \qquad (17)
\end{aligned}$$

Bounds for the remaining flux densities are as follows,

$$\begin{aligned}
F^u_{x_i,R} &\le C\frac{R_0}{R}\int_0^1 ||u||_2||u\phi^{2\rho-3/2}||_2^{1/2}||\nabla(u\phi^{1/2})||_2^{3/2}\,dx\,dt\\
&\le C\Big(\frac{R_0}{R}\Big)^4 \sup_t ||u||_2^4\int_0^1 ||u\phi^{2\rho-3/2}||_2^2\,dt + \frac{1}{8Re}\int_0^1 ||\nabla(u\phi^{1/2})||_2^2\,dt\\
&\le C\big(Re^{4/3}\sup_t ||u||_2^4+1\big)\frac{1}{Re}\Big(\frac{R_0}{R}\Big)^4\int_0^1 ||u\phi^{2\rho-3/2}||_2^2\,dt\\
&\quad + \frac{1}{4Re}\int_0^1 ||(\nabla u)\phi^{1/2}||_2^2\,dt,
\end{aligned}$$

and,

$$\begin{aligned}
F^b_{x_i,R},\ F^{ub}_{x_i,R} &\le C\big(Rm^{4/3}\sup_t ||u||_2^4+1\big)\frac{S}{Rm}\Big(\frac{R_0}{R}\Big)^4\int_0^1 ||b\phi^{2\rho-3/2}||_2^2\,dt\\
&\quad + \frac{S}{4Rm}\int_0^1 ||(\nabla b)\phi^{1/2}||_2^2\,dt.
\end{aligned} \tag{18}$$

Henceforth, we will be concerned with the scenario in which the kinetic energy over the integral domain and within the prescribed time-scale remains bounded away from zero. A quick comment on the (spatial) scaling of the non-dimensional flow is necessary to give a proper interpretation of the following results. Because in the dimensionless context u scales like 1 it is reasonable to assume $e_0^u > 1$. In the dimensional context we would equivalently assume that $e_0^u > R_0^2T^{-2}$. This assumption will be necessary in order to establish cascades for fluxes which exclude the fluid pressure flux.

We are now ready to establish the cascade for the direct kinetic to kinetic energy transfer driven by the advection of the velocity field.

Theorem 12. *Assume u and b are solutions of 3D MHD possessing sufficient regularity so that the generalized energy equality holds. Let $\{x_i\}_{i=1}^n \subset B(x_0,R_0)$ be the centers of a (K_1,K_2)-cover at scale R. Suppose that the flow is such that $e_0^u > 1$. For certain values β_u and β_b, if*

$$\tau := \max\left\{\Big(\frac{e_0^u}{E_0^u}\Big)^{1/4}, \Big(\frac{e_0^b}{E_0^b}\Big)^{1/4}\right\} < \min\{\beta_u,\beta_b\} =: \beta,$$

then, for scales R where $\tau/\beta \le R \le R_0$, we have,

$$\frac{1}{2K_1}\Big(\frac{1}{Re}E_0^u + \frac{S}{Rm}E_0^b\Big) \le \langle F^u\rangle_R \le 2K_2\Big(\frac{1}{Re}E_0^u+\frac{S}{Rm}E_0^b\Big).$$

Proof. The main estimates have already been established. Based on the generalized energy equality we have

$$\begin{aligned}
F^u_{x_i,R} \ge \frac{1}{Re}E^u_{x_i,R} + \frac{S}{Rm}E^b_{x_i,R} &- \big|F^p_{x_i,R}+N_{x_i,R}\big|\\
&-\frac{1}{2}\int_0^1\int |u|^2\phi_t\,dx\,dt - \frac{1}{2Re}\int_0^1\int |u|^2\Delta\phi\,dx\,dt\\
&-\frac{S}{2}\int_0^1\int |b|^2\phi_t\,dx\,dt - \frac{S}{2Rm}\int_0^1\int |b|^2\Delta\phi\,dx\,dt.
\end{aligned}$$

By the assumption on the flow we can replace the exponent of $1/9$ in the bound (17) with 1. Taking ensemble averages, we obtain the lower bound,

$$\langle F^u\rangle_R \geq \frac{1}{2K_1 Re} E^u_{x_i,R} - C_u K_2 \frac{1}{Re}\left(\frac{R_0}{R}\right)^4 e_0^u + \frac{S}{K_1 Rm} E^b_{x_i,R} - C_b K_2 \frac{S}{Rm}\left(\frac{R_0}{R}\right)^4 e_0^b,$$

where

$$C_u = C\big(C_p Re^{4/3} + Re + 1 + (MRm)^2 \sup_t ||u||_2^2 \sup_t ||b||_2^2\big),$$

and,

$$C_b = C\big(Rm^{4/3} \sup_t ||u||_2^4 + Rm + 1\big).$$

This is sufficient to establish values for β_u and β_b (containing no gradients) and conclude in the standard fashion. □

At this point, noting that F^b and F^{ub} both satisfy (9), we have already demonstrated all the steps involved in proving the existence of cascades for the densities F^b and F^{ub}. We consequently omit the proofs. The existence theorem for the cascades of the associated energy transfers follows.

Theorem 13. *Let $\{x_i\}_{i=1}^n \subset B(x_0, R_0)$ be the centers of a (K_1, K_2)-cover at scale R. Suppose that the flow is such that $e_0^u \geq 1$. For certain values β_u and β_b, if*

$$\tau := \max\left\{\left(\frac{e_0^u}{E_0^u}\right)^{1/4}, \left(\frac{e_0^b}{E_0^b}\right)^{1/4}\right\} < \min\{\beta_u, \beta_b\} =: \beta,$$

then for scales R where $\tau/\beta \leq R \leq R_0$, we have,

$$\frac{1}{2K_1}\left(\frac{1}{Re}E_0^u + \frac{S}{Rm}E_0^b\right) \leq \langle F^b, F^{ub}\rangle_R \leq 2K_2\left(\frac{1}{Re}E_0^u + \frac{S}{Rm}E_0^b\right).$$

7. Locality of the flux

According to turbulence phenomenology in the purely hydrodynamical setting, the average energy flux at scale R is supposed to be *well-correlated* only with the average fluxes at *nearby scales* (throughout the inertial range). This phenomenon has been confirmed in [16, 8, 10, 11].

In the plasma setting, the question of locality has been somewhat controversial. Recently, Aluie and Eyink [1] produced an argument in favor of locality of the total energy and cross-helicity fluxes. Numerical work also supports locality (c.f. [15] for the case of decaying turbulence).

In the present setting, locality of the flux is a simple consequence of existence of the corresponding nearly-constant turbulent cascade *per unit mass*.

We illustrate this on the example of the kinetic energy flux (transported by the velocity).

Denote the time-averaged local fluxes associated to the cover element $B(x_i, R)$ by $\hat{\Psi}_{x_i,R}$,

$$\hat{\Psi}_{x_i,R} = \frac{1}{T}\int_0^T \int \frac{1}{2}|u|^2 (u\cdot\nabla\phi_i)\, dx,$$

and the time-averaged local fluxes associated to the cover element $B(x_i, R)$, *per unit mass*, by $\hat{\Phi}_{x_i,R}$,

$$\hat{\Phi}_{x_i,R} = \frac{1}{T}\int_0^T \frac{1}{R^3}\int \frac{1}{2}|u|^2 (u\cdot\nabla\phi_i)\, dx.$$

Then, the (time and ensemble) averaged flux is given by

$$\langle\Psi\rangle_R = \frac{1}{n}\sum_{i=1}^{n}\hat{\Psi}_{x_i,R} = R^3\,\langle\Phi\rangle_R = R^3\frac{1}{n}\sum_{i=1}^{n}\hat{\Phi}_{x_i,R}.$$

The following manifestation of locality follows directly from Theorem 12.

Let R and r be two scales within the inertial range delineated in the theorem. Then

$$\frac{1}{4K_1^2}\left(\frac{r}{R}\right)^3 \le \frac{\langle\Psi\rangle_r}{\langle\Psi\rangle_R} \le 4K_2^2\left(\frac{r}{R}\right)^3.$$

In particular, if $r = 2^k R$ for some integer k,

$$\frac{1}{4K_1^2}\,2^{3k} \le \frac{\langle\Psi\rangle_{2^kR}}{\langle\Psi\rangle_R} \le 4K_2^2\,2^{3k},$$

i.e., along the *dyadic scale*, the locality propagates exponentially.

8. A Scenario Exhibiting Predominant u-to-b Energy Transfer.

In [13] a dynamic estimate is given on the vortex-stretching term (in the vorticity formulation of 3D NSE) – across a range of scales – using the ensemble averaging process we have illustrated above. The purpose was to present a mathematical evidence of the creation and persistence of integral scale length vortex filaments by establishing positivity of the ensemble-averaged vortex stretching term across a range of scales extending to the integral scale.

In the induction equation for the magnetic field, the nonlinear term $(b\cdot\nabla)u$ is responsible for the stretching of magnetic field lines. Positivity of $(b\cdot\nabla)u\cdot(\phi\, b)$ indicates the magnetic field line is being elongated, a phenomenon which corresponds to a transfer of energy from the velocity field to the magnetic field (negativity would reflect a diminution of the field line and a local transport of energy from the magnetic field to the fluid flow). Consequently, to conclude that the predominant energy exchange between the velocity and magnetic fields is from the velocity field to the magnetic field across a range of physical scales, it will be sufficient to establish (in an appropriate statistical sense) the positivity of $(b\cdot\nabla)u\cdot(\phi\, b)$ across these scales. Before proceeding to this task we remark that recent work (c.f. [15]) indicates that imbalanced exchanges are common in certain forced and decaying turbulent regimes.

We label the space-time localized quantity of interest as,

$$V_{x_i,R} = \int_0^1\int (b\cdot\nabla)u\cdot(b\,\phi_{x_i,R})\,dx\,dt.$$

Theorem 14. *Let $\{x_i\}_{i=1}^n \subset B(x_0,R_0)$ be the centers of a (K_1,K_2)-cover at scale R. For a certain value, β, if*

$$\tau := \left(\frac{e_0^b}{E_0^b}\right)^{1/4} < \beta,$$

then for scales R where $\tau/\beta \le R \le R_0$, we have,

$$\frac{1}{2K_1}E_0^b \le \langle V_{x_i,R}\rangle_R \le 2K_2E_0^b.$$

Proof. Starting with the dimensionless induction equation (with $Rm = S = 1$) and assuming b, is regular it is routine to obtain,

$$\int_0^1 \int (b \cdot \nabla) u \cdot (b\phi_{x_i,R})\, dx\, dt = \int_0^1 \int |\nabla b|^2 \phi_{x_i,R}\, dx\, dt - \int_0^1 \int \frac{1}{2} |b|^2 (\partial_t \phi_{x_i,R} - \Delta \phi_{x_i,R})\, dx\, dt + \frac{1}{2} \int_0^1 \int |b|^2 (u \cdot \nabla \phi_{x_i,R})\, dx\, dt.$$

Note that for a refined cut-off function ϕ the bounding process evident in the derivation of (9) can be modified to yeild,

$$\int_0^1 \int \frac{|b|^2}{2} u \cdot \nabla \phi\, dx\, dt \leq C \left(\frac{R_0}{R}\right)^4 \big(\sup_t ||u||_2\big)^4 \int_0^1 ||b\phi||_2^2\, dt + \frac{1}{4} \int_0^1 ||\nabla(\phi\, b)||_2^2\, dt.$$

Consequently, after taking ensemble averages and recalling the last two estimates in (8),

$$\langle V_{x_i,R} \rangle_R \leq K_2 E_0^b + K_2 \big(\sup_t ||u||_2^4 + 2\big) \left(\frac{R_0}{R}\right)^2 e_0^b$$

and,

$$\langle V_{x_i,R} \rangle_R \geq K_2 E_0^b - \frac{C}{2K_1} \big(\sup_t ||u||_2^4 + 2\big) \left(\frac{R_0}{R}\right)^2 e_0^b.$$

Selecting an appropriate value for β allows us to conclude in the standard fashion. □

9. A Note on 3D Incompressible Non-Decaying Fluid Turbulence.

Existence and locality of the kinetic energy cascade in 3D incompressible *decaying* (no external force, global kinetic energy decaying to zero) fluid turbulence in physical scales has been recently obtained in [10, 11].

This was based on existence and locality of the cascade featuring ensemble averaged fluxes of time-averaged, cover elements-localized *total energy* (kinetic plus potential),

$$\frac{1}{T} \int_0^T \int \left(\frac{|u|^2}{2} + p\right) (u \cdot \nabla \phi)\, dx\, dt.$$

The total energy cascade is universal, i.e., independent of whether global (or integral) kinetic energy is decaying or not; the assumption of decaying turbulence was used to obtain a *bona fide* kinetic energy cascade, as postulated in the turbulence phenomenology.

The estimates on the pressure from Section 6 allow us to now consider the opposite scenario, i.e., the scenario in which the kinetic energy associated with the integral domain is bounded away from zero – the case of *non-decaying* turbulence.

Henceforth, we consider suitable weak solutions to the 3D, dimensionless Navier-Stokes equations where, for convenience, $Re = 1$. We further assume the solutions possess sufficient regularity to ensure no loss of energy due to potential singularities is evident – i.e., the generalized energy equality holds. From this equality we obtain a formula for the flux (localized to the support of the

refined cut-off function ϕ) of interest:

$$F_\phi = \int_0^1 \int \frac{|u|^2}{2}(u \cdot \nabla\phi)\, dx\, dt = \int_0^1 \int |\nabla u|^2 \phi\, dx\, dt - \int_0^1 \int \frac{|u|^2}{2}\big(\partial_t \phi + \Delta\phi\big)\, dx\, dt$$
$$- \int_0^1 \int p(u \cdot \nabla\phi)\, dx\, dt.$$

To proceed note that the bounds from the MHD context on the energy-level localized quantities on the right hand side of the above equation all hold. For convenience, recall that these were,

$$\frac{1}{2}\int_0^1 \int |u|^2 \phi_t\, dx\, dt \le c_0 \left(\frac{R_0}{R}\right)^4 \int_0^1 \int |u|^2 \phi^{4\rho-3}\, dx\, dt,$$
$$\frac{1}{2}\int_0^1 \int |u|^2 \Delta\phi\, dx\, dt \le c_0 \left(\frac{R_0}{R}\right)^4 \int_0^1 \int |u|^2 \phi^{4\rho-3}\, dx\, dt,$$
$$\int_0^1 \int p(u \cdot \nabla\phi)\, dx\, dt \le C_p \left(\frac{R_0}{R}\right)^4 \left(\int_0^1 ||u\phi^{4\rho-3}||_2^2\, dt\right)^{1/9} + C\left(\frac{R_0}{R}\right)^4 \int_0^1 ||u\phi^{4\rho-3}||_2^2\, dt$$
$$+ \frac{1}{4}\int_0^1 ||(\nabla u)\phi^{1/2}||_2^2\, dt,$$

where C_p is as in (16). The ensemble average of the sublinear term in the above pressure bound is still treated as in (17). That is, upon ensemble averaging localized (by refined cut-off functions associated to elements of a (K_1, K_2)-cover of $B(x_0, R_0)$) dimensionless space-time averages, we have,

$$\frac{C_p}{n}\sum_{i=1}^n \frac{R_0^3}{R^3}\left(\frac{R_0}{R}\right)^{4/3}\left(\int_0^1 \int |u|^2 \phi_{x_i,R}\, dt\right)^{1/9} \le C_p\left(\frac{R_0}{R}\right)^4 (K_2 e_0)^{1/9} \le C_p \left(\frac{R_0}{R}\right)^4 K_2 e_0,$$

where in the last step we imposed an assumption on the flow motivated by the scaling properties of solutions to the dimensionless 3D NSE, namely that $e_0 \ge 1$. Ensemble averaging then gives the following interpolative bounds,

$$\frac{3}{4K_1}E_0 - C_u K_2 \left(\frac{R_0}{R}\right)^4 e_0 \le \langle F\rangle_R \le \frac{5}{4}K_2 E_0 + C_u K_2\left(\frac{R_0}{R}\right)^4 e_0,$$

where,

$$C_u = C_p + C + 2c_0.$$

The correction to the inertial range is now specified to be,

$$\beta = \left(\frac{1}{2C_u K_1 K_2}\right)^4.$$

Provided then that,

$$\tau := \left(\frac{e_0}{E_0}\right)^{1/4} \le \beta\frac{R}{R_0},$$

our interpolation becomes, for an appropriate constant K,

$$\frac{1}{2K}E_0 \le \langle F\rangle_R \le 2KE_0.$$

Remark 15. *The inertial range-correction parameter β depends (via C_p) on global* a priori *bounded quantities featuring no dependence on ∇u ($L_t^\infty L_x^2$ and $L_t^3 L_x^3$ norms of u). Consequently, the inertial range condition is still a requirement that the gradient of the velocity is large (space-time averaged, over the integral domain) with respect to the velocity itself; this will hold provided the integral domain is a region exhibiting high spatial complexity of the flow.*

We summarize our finding in the following theorem.

Theorem 16. *Given the definitions and labelings above, that the flow satisfies* $e_0 \geq 1$, *and provided* R *lies within the inertial range, i.e.,*

$$\frac{\tau}{\beta} \leq \frac{R}{R_0} \leq 1,$$

we have,

$$\frac{1}{2K} E_0 \leq \langle F \rangle_R \leq 2K E_0.$$

Remark 17. *The above exposed approach opens up an avenue for the study of* forced turbulence *in* physical scales *of 3D incompressible flows. This is the topic of the current research project* [5].

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